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TWO-DIMENSIONAL FLOWS OF A RELAXING MIXTURE AND THE STRUCTURE OF WEAK SHOCK WAVES

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Plane-parallel and axisymmetric flows of a chemically active mixture in which only a single reaction takes place are considered on the assumption that the equilibrium and the frozen speeds of sound in the medium are nearly equal. The asymptotic system of equations which in the nonlinear theory of small perturbations is valid in the range of transonic speeds is used. An exact particular solution of these equations is derived, which makes it possible to trace the process of shock wave onset and development. If the particle velocity is higher than the equilibrium but lower than the frozen speeds of sound, the shock waves are totally dispersed, as in the case of one-dimensional flows. Waves containing discontinuities with incomplete dispersion are generated, if the stream velocity exceeds the frozen speed of sound.

Flows of chemically active gas mixture in which the equilibrium and the frozen speeds of sound were close to each other were first considered by Napolitano [1]. Using the nonlinear theory of small perturbations and assuming that at any point of space the particle velocity does not much differ from both of these speeds of sound, he derived an asymptotic system of equations which is satisfied by the parameters of the mixture. Further development of Napolitano's method appears in [2, 3].

1. The strictly transonic mode. We assume, as in [1–3], that only a single reaction takes place in the mixture whose composition is defined by the single parameter q called the completeness of reaction. We denote the specific volume, entropy and internal energy by V, s and e, respectively, and the affinity and rate of the chemical reaction by Q and $\dot{q}$, respectively. In this notation the expressions for thermodynamic derivatives are

$$e_{11} = \left(\frac{\partial^2 e}{\partial q^2}\right)_{V,s}, \quad e_{12} = \left(\frac{\partial^2 e}{\partial q \partial V}\right)_s, \quad H_1 = -\left(\frac{\partial \dot{q}}{\partial Q}\right)_{q,V}$$

We define the difference between the equilibrium a_e and the frozen a_f speeds of sound by a small parameter ε_a, and denote the basic equilibrium state of gas on which small perturbations are imposed, by subscript ∞. The particle velocity modulus and the pressure of gas are denoted, respectively, by v and p. In conformity with [1–3] for a strictly transonic mode in which the particle velocity at every point of space is close to both speeds of sound, we have

$$e_{12\,\infty} = \varepsilon_a \frac{p_\infty}{q_\infty} e'_{12\,\infty}, \quad v_\infty - a_{e\infty} = \varepsilon_a^2 v_\infty \sigma_{e\infty}, \quad v_\infty - a_{f\infty} = \varepsilon_a^2 v_\infty \sigma_{f\infty} \quad (1.1)$$

where the order of magnitude of dimensionless constants $e'_{12\infty}$, $\sigma_{e\infty}$ and $\sigma_{f\infty}$ is equal unity.

Let v_x and v_r be projections of the velocity vector on the x- and r-axes of a Cartesian or cylindrical system of coordinates, L be a characteristic dimension along the x-axis, and ε and Δ denote small parameters. We introduce dimensionless variables defined by formulas

$$x = Lx', \quad r = \frac{L}{\Delta} r', \quad v_x = v_\infty (1 + \varepsilon v_x'), \quad v_r = \varepsilon \Delta v_\infty v_r' \quad (1.2)$$

The deviations of density $\rho = 1/V$, pressure p, and of equilibrium a_e and frozen a_f speeds of sound from their related values in the basic uniform stream are proportional to ε. Thus

$$\rho = \rho_\infty (1 + \varepsilon \rho'), \quad p = p_\infty (1 + \varepsilon p') \quad (1.3)$$

$$a_e = a_{e\infty} (1 + \varepsilon a_e'), \quad a_f = a_{f\infty} (1 + \varepsilon a_f')$$

and the perturbed completeness of reaction, chemical affinity and the rate of reaction must be of the order of the product of small parameters ε and ε_a

$$q = q_\infty (1 + \varepsilon \varepsilon_a q'), \quad Q = \varepsilon \varepsilon_a \frac{p_\infty}{q_\infty \rho_\infty} Q', \quad \dot{q} = \varepsilon \varepsilon_a \frac{q_\infty}{\tau} \dot{q}' \quad (1.4)$$

where τ is the relaxation time. We introduce two more dimensionless quantities

$$N_r = \frac{L}{\tau v_\infty}, \quad H_1 = \frac{q_\infty^2 \rho_\infty}{\tau p_\infty} H_1'$$

related to that time.

It remains to substitute formulas (1.1) – (1.4) into the equations to which motions of the relaxing mixture are subordinated. We retain in all equations only the principal terms, neglecting terms of higher order of smallness, and follow the asymptotic analysis presented in [3]. For convenience we omit the primes at all dimensionless quantities and define small parameters by the following relationships:

$$2\varepsilon m_\infty = \Delta^2 = 2\varepsilon_a{}^2 \frac{p_\infty^2 e_{12\,\infty}^2}{q_\infty^2 \rho_\infty^2 e_{11\,\infty} v_\infty^2}$$

with

$$\sigma = \frac{\varepsilon_a{}^2}{\varepsilon}\frac{\sigma_{e\infty}}{m_\infty}, \qquad l = \frac{1}{N_r}\frac{p_\infty}{q_\infty^2 \rho_\infty e_{11\,\infty} H_{1\infty}}$$

In the considered approximation we have for the thermodynamic coefficient

$$m_\infty = m_{e\infty} = m_{f\infty} = \frac{1}{2\rho_\infty^3 v_\infty^2}\left(\frac{\partial^2 a}{\partial V_\infty^2}\right)_{q,\,s} = \frac{1}{2\rho_\infty^3 v_\infty^2}\left(\frac{\partial^2 p}{\partial V_\infty^2}\right)_{Q,\,s}$$

Using the results given in [3], we represent the system of asymptotic equations defining the perturbed nonequilibrium flow of mixture in the form

$$\left(v_x + \sigma - \frac{1}{2}\right)\frac{\partial v_x}{\partial x} - \left[\frac{\partial v_r}{\partial r} + (\nu - 1)\frac{v_r}{r}\right] = \frac{1}{2}\frac{q_\infty^2 \rho_\infty e_{11\,\infty}}{p_\infty e_{12\,\infty}}\frac{\partial q}{\partial x} \tag{1.5}$$

$$\frac{\partial q}{\partial x} = -\frac{1}{l}\left(q + \frac{p_\infty e_{12\,\infty}}{q_\infty^2 \rho_\infty e_{11\,\infty}} v_x\right), \qquad \frac{\partial v_x}{\partial r} = \frac{\partial v_r}{\partial x}$$

In the case of plane-parallel motions the parameter $\nu = 1$, while in that of axisymmetric motion $\nu = 2$. The system of quasi-linear equations (1.5) is for $v_x > 1/2 - \sigma$ of the hyperbolic kind.

Since the remainder

$$\sigma_{f\infty} - \sigma_{e\infty} = -\frac{1}{2}\frac{p_\infty^2 e_{12,\infty}^2}{q_\infty^2 \rho_\infty^2 e_{11\,\infty} v_\infty^2}$$

this condition means that the velocity of gas particles exceeds the local frozen speed of sound. Directions of the three characteristics are defined at every point by the relationship

$$\frac{dx}{dr} = \pm\sqrt{v_x + \sigma - \frac{1}{2}}, \qquad \frac{dr}{dx} = 0 \tag{1.6}$$

Introducing in the analysis the new unknown function $u = v_x + \sigma$, from system (1.5) we obtain for its determination the single third order equation

$$\frac{1}{2}\frac{\partial^2 u^2}{\partial x^2} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{\nu - 1}{r}\frac{\partial u}{\partial r}\right) = \frac{l}{2}\frac{\partial^3 u}{\partial x^3} - l\frac{\partial}{\partial x}\left[\frac{1}{2}\frac{\partial^2 u^2}{\partial x^2} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{\nu - 1}{r}\frac{\partial u}{\partial r}\right)\right] \tag{1.7}$$

containing only one constant parameter l, which depends on the properties of the relaxing mixture.

2. Transformation to an ordinary differential equation. Equation (1.7) is more convenient for further operations than system (1.5). We seek its solution in the form

$$u = 4\left(\frac{d}{c}\right)^2 r^2 - 2\nu\frac{d}{c^2} f(\xi), \qquad \xi = cx - dr^2 \tag{2.1}$$

where c and d are arbitrary constants and function $f(\xi)$ is taken to be the integral of the third order ordinary differential equation

$$f\frac{d^2 f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 - \frac{df}{d\xi} - \frac{2}{\nu} = -lc\left[f\frac{d^3 f}{d\xi^3} + \left(3\frac{df}{d\xi} - 1\right)\frac{d^2 f}{d\xi^2}\right] - \frac{lc^3}{4\nu d}\frac{d^3 f}{d\xi} \tag{2.2}$$

Solutions of form (2.1) were often used for defining flows of inert gas. A stream with local supersonic zones at the walls of a Laval nozzle was investigated in [4, 5] on the assumption of absence of dissipative processes. The use of formulas (2.1) for determining the velocity field of an inviscid gas in another off-design mode in a nozzle with the supersonic region occupying the whole throat and containing downstream a compression shock was shown in [6]. The possibility of extending these solutions to the analysis of motions of a viscous and heat-conducting gas of both kinds was indicated in [7, 8]. The transformation of a dissipating gas stream into a perfect inviscid flow was investigated in [9] by means of passing to limit.

It will be evident from the subsequent analysis that integral (2.1) with function $f(\xi)$ satisfying Eq. (2.2) makes it possible to trace the onset and development of shock waves generated in flows of a chemically active mixture.

The integration of Eq. (2.2) yields

$$\alpha(\beta+f)\frac{d^2f}{d\xi^2}-(\alpha-f)\frac{df}{d\xi}+\alpha\left(\frac{df}{d\xi}\right)^2-f-\frac{2}{\nu}\xi=\frac{2}{\nu}\alpha A \qquad (2.3)$$

$$\left(\alpha=lc, \quad \beta=\frac{c^2}{4\nu d}\right)$$

where A is an arbitrary constant. For convenience we assume that constants c and d are positive, which implies that constants α and β are also positive.

To investigate the properties of the nonlinear equation (2.3) we first determine the form of functions q and v_r. Substituting the expression

$$\frac{q_\infty^2\rho_\infty e_{11\,\infty}}{p_\infty e_{12\,\infty}}q=\sigma-4\left(\frac{d}{c}\right)^2r^2+2\nu\frac{d}{c^2}h(\xi) \qquad (2.4)$$

into the second equation of system (1.5), we obtain

$$\alpha\frac{dh}{d\xi}+h=f$$

Eliminating from the analysis function h which for $\xi\to-\infty$ increases exponentially, we obtain from this

$$h=\frac{1}{\alpha}\exp\left(-\frac{\xi}{\alpha}\right)\int\exp\left(\frac{\xi}{\alpha}\right)f\,d\xi \qquad (2.5)$$

Let us determine the transverse component of the velocity vector. The last equation of system (1.5) implies that

$$v_r=B_1r+B_2r^3+8\left(\frac{d}{c}\right)^2xr+4\nu\frac{d^2}{c^3}rf(\xi) \qquad (2.6)$$

It remains to determine constants B_1 and B_2. To do this we use the different expressions for the derivative $\partial q/\partial x$ which are obtained from the first and second equations of system (1.5). The substitution into the right-hand side of the second of these equations formulas (2.1) and (2.4) for $v_x=u-\sigma$ and q, respectively, yields

$$\frac{q_\infty^2\rho_\infty e_{11\,\infty}}{p_\infty e_{12\,\infty}}\frac{\partial q}{\partial x}=-2\nu\frac{d}{lc^2}(h-f)$$

Taking into account formula (2.6), from the first equation of system (1.5) we have

$$\frac{q_\infty^2\rho_\infty e_{11\,\infty}}{p_\infty e_{12\,\infty}}\frac{\partial q}{\partial x}=2\left[-\nu B_1-(2+\nu)B_2r^2-8\nu\left(\frac{d}{c}\right)^2x-4\nu^2\frac{d^2}{c^3}f+\right.$$
$$\left.\nu\frac{d}{c}\frac{df}{d\xi}+4\nu^2\frac{d^2}{c^3}f\frac{df}{d\xi}\right]$$

where the right-hand side must depend on the combination of ξ independent variables of x and r. This condition is satisfied only for

$$B_2 = -\frac{8\nu}{2+\nu}\left(\frac{d}{c}\right)^3$$

Equating the above two expressions for $\partial q / \partial x$, we obtain

$$\beta h = \alpha\beta\frac{c}{d}B_1 + 2\frac{\alpha}{\nu}\xi + (\alpha+\beta) f - \alpha(\beta+f)\frac{df}{d\xi} \tag{2.7}$$

Recalling formula (2.5) for function h and differentiating expression (2.7), we obtain Eq. (2.3) with $(2\alpha d + \nu\beta c B_1)/(\nu d)$ in its right-hand side. Hence it is obvious that

$$B_1 = 8l\left(\frac{d}{c}\right)^2(A-1)$$

Let us also note the form of function Q, which defines the extent of deviation of the state of mixture from complete thermodynamic equilibrium. It was shown in [3] that in the considered approximation

$$Q = \frac{q_\infty^2 \rho_\infty e_{11\,\infty}}{p_\infty} q + e_{12\,\infty} v_x = 2\nu e_{12\,\infty}\frac{d}{c^2}(h-f) \tag{2.8}$$

Let us revert to Eq. (2.3). The following two particular integrals play an important part in the analysis of its solutions:

$$f = a_{1,2}\xi + \alpha b_{1,2}, \quad a_{1,2} = \frac{1}{2}\left(1 \mp \sqrt{1+\frac{8}{\nu}}\right), \quad b_{1,2} = \frac{2}{\nu}\frac{A-1}{a_{1,2}-1} \tag{2.9}$$

The variation of the chemical reaction rate affects coefficient l and with it also α. For $\alpha = 0$ formulas (2.9) define the motions of an inviscid inert gas [4 – 6]. However the presence in the definition of function f of a term proportional to α is unimportant, since it can be made to vanish by simply altering in the input equations (1.5) and (1.7) the origin of the x-coordinate. This can be also achieved by setting constant $A = 1$, when $b_{1,2}$ and constant B_1 vanish, which corresponds to the case when the velocity field of an inert gas flow is analyzed. Let us set $A = 1$. Then the integral $f = a_1\xi$ represents a stream whose velocity field at the inlet of a Laval nozzle is subsonic and in the neighborhood of the nozzle critical cross section passes through both the equilibrium and the frozen speed of sound.

Let us derive the initial conditions for integrating Eq. (2.3). The solution which determines a chemically active mixture must obviously be at infinity upstream, i.e. for $\xi \to -\infty$, close to the solution for an inviscid inert gas. However all operation modes of a nozzle through which flows an inert gas are defined by relationships where for $\xi \to -\infty$ tend to integral (2.9) with $a = a_1$ [4 – 6]. It follows from this that initial data for the integration of Eq. (2.3) must be chosen for considerable negative values of ξ in the neighborhood of $f = a_1\xi$. To obtain a more exact idea of the asymptotic behavior of the sought solution we set

$$f = a\xi + \chi(\xi), \quad a = a_1 \tag{2.10}$$

and, assuming that the quantity $\chi(\xi)$ is small in comparison with $a\xi$, we linearize Eq. (2.3) and obtain

$$\alpha(\beta + a\xi)\frac{d^2\chi}{d\xi^2} + [\alpha(2a-1) + a\xi]\frac{d\chi}{d\xi} + (a-1)\chi = 0$$

After the substitution of $\eta = -(\beta + a\xi)/(\alpha a)$ for the independent variable, the last equation can be written as

$$\eta \frac{d^2\chi}{d\eta^2} + \left(\frac{2a-1}{a} - \frac{\beta}{\alpha a} - \eta\right)\frac{d\chi}{d\eta} - \frac{a-1}{a}\chi = 0$$

which is the canonical form of the confluent hypergeometric equation [10]. Using the conventional notation for such equations, we write its solution as

$$\chi = C_1\Phi\left(\frac{a-1}{a},\ \frac{2a-1}{a} - \frac{\beta}{\alpha a};\ \eta\right) + C_2\eta^{-\frac{a-1}{a}+\frac{\beta}{\alpha a}}\ \Phi\left(\frac{\beta}{\alpha a},\ \frac{1}{a} + \frac{\beta}{\alpha a};\ \eta\right) \quad (2.11)$$

It remains to determine the relation between constants C_1 and C_2. This can be achieved by using the asymptotic form of hypergeometric functions for $\eta \to +\infty$. For $\alpha > 0$ and $\xi \to -\infty$ the variable $\eta \to +\infty$. Hence [10]

$$\chi = \eta^{-1+\frac{\beta}{\alpha a}} e^{\eta} G\left(\frac{1}{a},\ 1 - \frac{\beta}{\alpha a};\ \eta\right)\left[C_1 \frac{\Gamma\left(\frac{2a-1}{a} - \frac{\beta}{\alpha a}\right)}{\Gamma\left(\frac{a-1}{a}\right)} + C_2 \frac{\Gamma\left(\frac{1}{a} + \frac{\beta}{\alpha a}\right)}{\Gamma\left(\frac{\beta}{\alpha a}\right)}\right] + \ldots$$

where Γ denotes Euler's gamma function, and $G(1/a,\ 1 - \beta/(\alpha a);\ \eta)$ represents an asymptotic series in inverse powers of η, which for $\eta \to \infty$ tends to unity. To obtain a solution which tends to vanish at infinity it is necessary to equate the expression in brackets to zero. This yields the relation between C_1 and C_2. Formula (2.11) now becomes

$$\chi = C_1\Big[\Phi\left(\frac{a-1}{a},\ \frac{2a-1}{a} - \frac{\beta}{\alpha a};\ \eta\right) - \frac{\Gamma\left(\frac{\beta}{\alpha a}\right)\Gamma\left(\frac{2a-1}{a} - \frac{\beta}{\alpha a}\right)}{\Gamma\left(\frac{a-1}{a}\right)\Gamma\left(\frac{1}{a} + \frac{\beta}{\alpha a}\right)}\eta^{-\frac{a-1}{a}+\frac{\beta}{\alpha a}}\Phi\left(\frac{\beta}{\alpha a},\ \frac{1}{a} + \frac{\beta}{\alpha a};\ \eta\right)\Big] \quad (2.12)$$

The linear combination of the hypergeometric functions appearing in brackets is proportional to the so-called Ψ-function [10], hence for $\eta \to +\infty$ and $\xi \to -\infty$ we obtain

$$\chi = \alpha^{-\frac{a-1}{a}} C\eta^{-\frac{a-1}{a}} + \ldots = C(-\xi)^{-\frac{a-1}{a}} + \ldots \quad (2.13)$$

$$C = \alpha^{\frac{a-1}{a}} C_1 \frac{\Gamma\left(\frac{\beta}{\alpha a}\right)}{\Gamma\left(-\frac{a-1}{a} + \frac{\beta}{\alpha a}\right)}$$

The constant C_1 in formulas (2.12) and (2.13) remains arbitrary and on it depend initial values of function $f(\xi)$. In the asymptotic expansion of $\chi(\xi)$ the exponent of the principal term is

$$-\frac{a-1}{a} = \frac{1 + \sqrt{1 + 8/\nu}}{1 - \sqrt{1 + 8/\nu}} < 0$$

hence the second term in the right-hand part of equality (2.10) is in fact considerably smaller than the first when $\xi \to -\infty$. It can also be shown that formula (2.13) represents the principal term of the asymptotic expansion of the correction $\chi(\xi)$ in the case of motion of an inert gas, which is governed by Eq. (2.3) with $\alpha = 0$.

3. Continuous flows. Let us elucidate the meaning of the particular value of $f = -\beta$ for which the coefficient at the leading derivative in the ordinary differential equation (2.3) vanishes. Recalling formulas (1.6) we write the equation which specifies the slope of characteristic curves

$$\left(\frac{dx}{dr}\right)^2 = 4\left(\frac{d}{c}\right)^2 r^2 - 2\nu\frac{d}{c^2}f(\xi) - \frac{1}{2} \tag{3.1}$$

We seek its solution in the form $\xi = \xi_c = \text{const}$, i.e. $x = c^{-1}(\xi_c + dr^2)$. It follows directly from Eq. (3.1) that $f(\xi_c) = -\beta$. Thus the intersection along any integral curve defined by Eq. (2.3) with the straight line $f = -\beta$ implies the intersection of a characteristic in the physical space.

Let us determine qualitatively the properties of the considered integral curves whose asymptotic behavior for $\xi \to -\infty$ is defined by expansion (2.13). First, we set the constant $C > 0$. It follows from formula

$$\frac{df}{d\xi} = a_1 + \frac{a_1 - 1}{a_1} C(-\xi)^{-\frac{2a_1-1}{a_1}} + \ldots$$

for the derivative that there exists an interval $-\infty < \xi < \xi_0$ in which $df/d\xi > a_1$. It can be readily shown that when condition $f(\xi_0) > -\beta$ is satisfied, the slope of the integral curve defined by Eq. (2.3) at point $\xi = \xi_0$ must remain greater than that of the straight line $f = a_1\xi$.

Let us assume the contrary, i.e. that $df(\xi_0)/d\xi = a_1$. Then $d^2f(\xi_0)/d\xi^2 \leqslant 0$. Furthermore, obviously

$$f(\xi_0) = a_1\xi_0 + \alpha b, \qquad b > 0 \tag{3.2}$$

Substituting these values of function f and of its first derivative into Eq. (2.3), we obtain

$$(\beta + f)\frac{d^2f}{d\xi^2} = ba_2 \tag{3.3}$$

The second derivative $d^2f(\xi_0)/d\xi^2$ determined by this equation is positive. The derived contradiction proves the above statement.

Let some integral curve defined by Eq. (2.3) intersect the straight line $f = -\beta$ at a point where $\xi = \xi_c$, and its slope and curvature remain finite. We have

$$\frac{df}{d\xi_c} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right) \pm \frac{1}{2}\sqrt{\left(1 - \frac{\beta}{\alpha}\right)^2 + \frac{8}{\nu}\left(1 + \frac{\xi_c}{\alpha}\right)} \tag{3.4}$$

The curve of derivative $df/d\xi_c$ is shown in Fig. 1, where

$$\xi_* = -\alpha\left[1 + \frac{\nu}{8}\left(1 - \frac{\beta}{\alpha}\right)^2\right], \qquad \frac{df_*}{d\xi_c} = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right)$$

We denote by ξ_1 the abscissa of the intersection point of straight lines $f = a_1\xi$ and $f = -\beta$. Obviously $\xi_1 = -\beta/a_1 > 0$. For $\xi_c = \xi_1$ from formula (3.4) we have

$$\left(\frac{df}{d\xi_1}\right)_1 = a_1, \qquad \left(\frac{df}{d\xi_1}\right)_2 = a_2 + \frac{\beta}{\alpha} \tag{3.5}$$

The first of these expressions for derivatives relates to the straight line $f = a_1\xi$.

Let us prove that the integral curve defined by Eq. (2.3) with asymptotics (2.13) cannot reach the straight line $f = -\beta$, when $\xi \to -\infty$ and $C > 0$. If it did intersect this line at some point where $\xi = \xi_c$, then by virtue of the previous proof we would have $\xi_c > \xi_1$ and $a_1 < df/d\xi_c \leqslant 0$. Since we have to take in Fig. 1 the lower branch of the curve shown there, where $\xi_c > \xi_1$, we find a contradiction with the initial estimate $df/d\xi_c < a_1$.

Let us now set in the asymptotic expansion (2.13) the constant $C < 0$. Then in some interval $-\infty < \xi < \xi_0$ the slope of the considered integral curve must be smaller than that of the straight line $f = a_1\xi$. If $f(\xi_0) > -\beta$, then also at point where $\xi = \xi_0$ the derivative $df(\xi_0)/d\xi < a_1$.

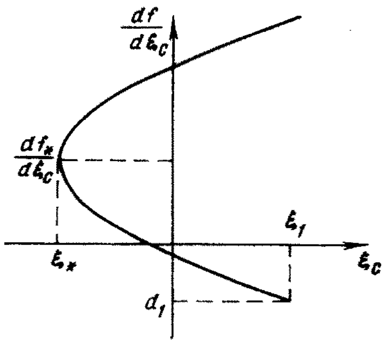

Fig. 1

As previously, the proof of this statement is derived by contradiction, i.e. by assuming that $df(\xi_0)/d\xi = a_1$. Then $d^2f(\xi_0)/d\xi^2 \geqslant 0$. Using the first of formulas (3.2), where the sign of constant b is to be reversed, we again obtain formula (3.3) with a negative right-hand part. This leads to a contradiction in the computation of $d^2f(\xi_0)/d\xi^2$.

The integral curve defined by Eq. (2.3) with asymptotics (2.13) and $C < 0$ can intersect the straight line $f = -\beta$ only at a right angle, since as just proved, at the intersection point $\xi_c < \xi_1$ and $df/d\xi_c < a_1$. Were the derivative $df/d\xi_c$ to remain finite, it would have to be defined by the lower branch of the curve shown in Fig. 1. However for $\xi_c < \xi_1$ that curve shows that $df/d\xi_c > a_1$, which implies that the slope of the considered integral curve becomes infinite when it reaches the straight line $f = -\beta$.

The infinitely great value of the derivative $df/d\xi_c$ for $f(\xi_c) = -\beta$ implies the onset in the physical space of infinitely great accelerations along the line which at each of its points has a characteristic slope. This line is the envelope of characteristic curves [11]. Hence the integral curves defined by Eq. (2.3) whose asymptotic behavior for $\xi \to -\infty$ is specified by expansion (2.13) cannot be used for the determination of fields of real flows.

Having established qualitatively the properties of Eq. (2.3), we can pass to its direct integration. Let us consider, as an example, the flow through a plane nozzle for which the constants are: $\nu = 1$ and $a_1 = -1$. Although the initial conditions for function f and its first derivative are defined by formulas (2.12) and (2.13), it is better to use the more exact asymptotic series

$$f = -\xi + C\frac{1}{\xi^2} - 3\alpha\beta C\frac{1}{\xi^4} - 2\left(4\alpha^2\beta - \frac{1}{3}C\right)C\frac{1}{\xi^5} + 15\alpha^2\beta(\beta - 2\alpha)C\frac{1}{\xi^6} + \ldots \tag{3.6}$$

$$\frac{df}{d\xi} = -1 - 2C\frac{1}{\xi^3} + 12\alpha\beta C\frac{1}{\xi^5} + 10\left(4\alpha^2\beta - \frac{1}{3}C\right)C\frac{1}{\xi^6} - 90\alpha^2\beta(\beta - 2\alpha)C\frac{1}{\xi^7} + \ldots$$

which are derived with the nonlinear terms in Eq. (2.3) taken into account. The results of computations are shown in Fig. 2. It was assumed that $\alpha = 0.1$, $\beta = 0.25$, and the coefficient C in formulas (3.6) was equal 10^{-1}, 10^{-2}, 10^{-3} and 10^{-6}. The integral (2.9) for $a = a_1$ and $a = a_2$ is represented by the straight lines ac and bd, respectively.

The values $v_x < -\sigma$ correspond to the interval in which particle velocity is lower than the frozen and equilibrium speeds of sound. For $-\sigma < v_x < {}^1/_2 - \sigma$ the equilibrium speed of sound limits the stream velocity from below and the frozen one provides its upper limit. The values $v_x > {}^1/_2 - \sigma$ correspond to the interval in which the particle velocity exceeds both speeds of sound. The regions in which the velocity of the stream is contained within one of the three intervals are denoted in Figs. 3 and 4 by letters A, B and C.

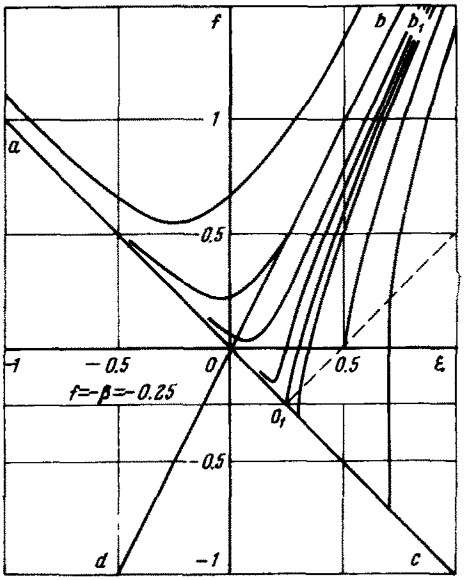

Fig. 2

The curves in the ξf-plane provide the distribution of the perturbed velocity $v_x(x, 0)$ along the central streamline (the nozzle axis), and in conformity with equalities (2.1)

$$v_x(x, 0) = -\sigma - 2\nu \frac{d}{c^2} f(\xi)$$

The curves in the half-plane $f > 0$ relate to flows in which particles move along the central streamline at a velocity lower than the equilibrium speed of sound, while in regions along the channel walls they can accelerate to velocities exceeding the equilibrium and the frozen speeds of sound.

It will be seen that for $f = -\beta$ function $v_x(x, 0) = {}^1/_2 - \sigma$. Hence the integral curves of Eq. (2.3) which intersect the band $-\beta < f < 0$ define flows at a velocity lower than the frozen but higher than the equilibrium speed of sound along the central streamline. The velocity of particles in such flows may exceed the first of the two speeds of sound at some distance from the axis of symmetry.

Both kinds of considered flows are continuous.

Let us compare these flows with one-dimensional flows of relaxing mixture considered in detail in [3, 12, 13]. The last of these papers contains some profound analysis of stability of such flows. Geometric characteristics of inert gas flow are given below for comparison.

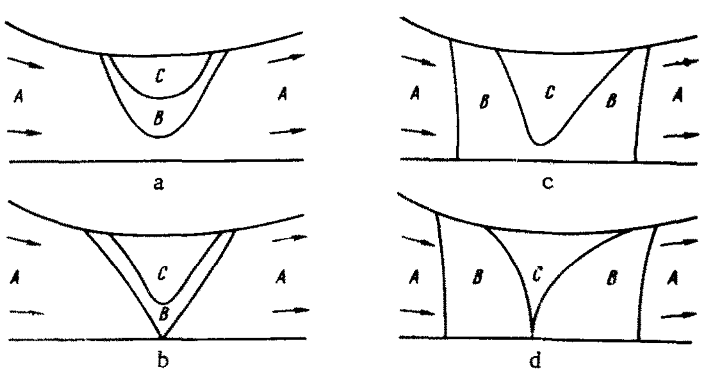

Fig. 3

Curves lying in the half-plane $f > 0$ represent flows which have no analogs among one-dimensional flows. These flows are diagrammatically shown in Fig. 3, a. At considerable distances from the nozzle axis the stream passes first through the equilibrium and then through the frozen speed of sound. If instead of a relaxing mixture an inert gas is considered, these flows correspond to flows with local supersonic zones adjacent to the channel walls.

The integral curve defined by Eq. (2.3) tangent to the axis $f = 0$ represents the limit flow in which regions where the particle velocity exceeds the equilibrium speed of sound merge at the central streamline (Fig. 3, b). The presence of a chemical reaction results in that the limit flow has no particular character. As in the case of a viscous inert gas [7 – 9], the derivative $\partial v_x(x, 0) / \partial x = 0$ at the point of junction of regions with $v_x(x, r) > -\sigma$. The curve along which particle velocity reaches the equilibrium speed of sound has a cusp at the axis of symmetry. The corresponding limit flow of an inviscid inert gas is shown in Fig. 2 by the broken line aob. In such flow the derivative $\partial v_x(x, 0) / \partial x \neq 0$ and is discontinuous at the junction point of local supersonic waves; only when the acoustic line is straight the acceleration vanishes at its intersection with the axis of symmetry [14, 15].

The stream field remains qualitatively unaltered when the perturbed velocity $v_x(x, 0)$ exceeds $-\sigma$. However for its maximum values close to the right-hand end of the interval $-\sigma < v_x(x, 0) < 1/2 - \sigma$ regions of relatively considerable gradients of the mixture parameters appear in the stream, as implied by the behavior of curves in Fig. 2. Such regions may be considered as shock waves with total dispersion, whose structure in one-dimensional flows was investigated in [3, 12, 13]. Although qualitatively the analogy is complete, however in the considered case the distribution of gas parameters inside the shock wave is somewhat more complicated, owing to the presence of the transverse component of the velocity vector. Flows for $-\sigma < v_x(x, 0) < 1/2 - \sigma$ are shown diagrammatically in Fig. 3, c. Passing from a relaxing mixture to an inert gas it is necessary to compare these with flows with local supersonic zones at the channel walls. A distinctive feature of relaxing mixture flows is that the curve along which particle velocity reaches the equilibrium speed of sound has two separate branches which connect the walls to the axis of symmetry. The neighborhood of the second of these branches, where the stream is rapidly decelerated, is a completely dispersed shock wave.

Let us now consider the second limit case, when the regions of velocities exceeding the frozen speed of sound merge at the central streamline. Such limit flow is represented in Fig. 2 by the line consisting of the straight ao_1 and the curve o_1b_1. This flow is of a particular kind since point o_1 corresponds to the characteristic of the input system of equations of gasdynamics along which various solutions interlock analytically. The region of considerable gradients represents here a completely dispersed shock wave upstream of which the particle velocity is exactly equal to the frozen speed of sound. A further property of the one-dimensional structure of such wave is the nonanalytic distribution of parameters of a relaxing mixture: the uniform stream is generally not subjected to the action of perturbations on one side of the point-characteristic, and is in the state of complete thermodynamic equilibrium [3, 12, 13]. At the merger of regions where $v_x(x, r) > 1/2 - \sigma$ the derivative $\partial v_x(x, 0) / \partial x \neq 0$ and is discontinuous. As noted above, the velocity field of an inviscid inert gas with local supersonic zones merging along the central streamline has a similar property. A consequence of this property is that the curve along which particle velocity reaches the frozen speed of sound intersects the axis of symmetry at a right angle and forms a cusp there (Fig. 3, d).

4. Incompletely dispersed shock waves. Let us consider another off-design operation mode of a Laval nozzle, when the whole central part of the channel is taken by the zone in which particles move at a speed which exceeds not only the equilibrium, but also the frozen speed of sound. In the ξf-plane the stream in the inlet part of the nozzle is represented by the straight line ao_1, hence the distribution of mixture parameters remains the same as in the previously described limit case of merger along the axis of symmetry of zones in which the mass flow rate is higher than the frozen speed of sound. Since the passing across the straight line $f = -\beta$ along line ac is admissible, the complete flow /upstream of and/ up to the shock front is defined by the integral (2.3) with $a = a_1 = -1$. Downstream of the compression shock the stream field is specified by function f which is determined by integrating Eq. (2.3).

Let us derive the boundary conditions which must be satisfied at the compression shock in the considered here approximation. We shall carry out the analysis of Hugoniot's equations by the method used by Busemann [16] in the case of transonic flows of an inert gas. We revert to the dimensional input equations and denote by subscript 1 the parameters of gas on the side of the discontinuity line which faces the oncoming stream and the related parameters on the opposite side of this line by subscript 2. The first condition

$$q_2 = q_1 \qquad (4.1)$$

is obvious; it is the consequence of the instantaneous shock compression of gas.

We denote by α_f the frozen Mach angle and by γ the angle between the tangent to the shock front and the axis r; in a strictly transonic mode the latter is small. For any weak shock wave, with allowance for equality (4.1), we have

$$\cos\gamma = \sin\alpha_{f_1}\left(1 + \frac{1}{2}\frac{m_{f_1}}{\sin^2\alpha_{f_1}}\frac{p_2 - p_1}{\rho_1 a_{f_1}{}^2}\right)$$

In computing the Mach angle it is necessary to take into consideration that the stream velocity at infinity is close to both speeds of sound. In the first approximation we have

$$\sin\alpha_{f_1} = \frac{a_{f_1}}{v_1} = 1 + m_{f\infty}\frac{p_1 - p_\infty}{\rho_\infty v_\infty^2} - \frac{v_\infty - a_{f\infty}}{v_\infty}$$

Expanding $\cos\gamma$ into a series, from the last two formulas we obtain

$$\gamma^2 = 2\frac{v_\infty - a_{f\infty}}{v_\infty} - m_\infty\frac{p_2 + p_1 - 2p_\infty}{\rho_\infty v_\infty^2} \qquad (4.2)$$

This formula relates the angle of the shock front slope to the change in pressure across the shock. The third condition which is to be satisfied at the compression shock is that of continuity of the tangential component of the velocity vector

$$\gamma v_{x1} + v_{r1} = \gamma v_{x2} + v_{r2} \qquad (4.3)$$

If $x = x_s(r)$ defines the discontinuity line, then $\gamma = dx_s / dr$. Using this equality, we substitute formulas (1.1) – (1.3) into conditions (4.2) and (4.3). It was shown in [1 – 3] that the pressure increase $p - p_\infty = -\rho_\infty v_\infty (v_x - v_\infty)$. Passing to dimensionless variables (omitting primes in superscripts), we now obtain

$$\left(\frac{dx_s}{dr}\right)^2 = \frac{1}{2}(u_2 + u_1 - 1), \qquad u_2\frac{dx_s}{dr} + v_{r2} = u_1\frac{dx_s}{dr} + v_{r1} \tag{4.4}$$

Let us assume that the shock front has the shape of a second order curve $\xi = \xi_s =$ const, i.e. $x = C^{-1}(\xi_s + dr^2)$. Taking this equality into account, from the first of conditions (4.4) we obtain

$$f_2 + f_1 = -2\beta \tag{4.5}$$

while the second condition is identically satisfied. The quantity $f_1 = a_1\xi_s$, hence the geometric locus of points which define the state of gas in the ξf-plane downstream of the shock front is a straight line, shown in Fig. 2 by the dash line.

Integration of Eq. (2.3) requires the prior determination of the derivative $df_2 / d\xi$. In the considered solution of equations of gasdynamics condition (4.1) has the form $h_2 = h_1$. This and expression (2.7) for function h yields

$$\frac{df_2}{d\xi} + \frac{df_1}{d\xi} = 2\left(1 + \frac{\beta}{\alpha}\right) \tag{4.6}$$

This formula is not valid in the limit case of $f_2 = f_1 = -\beta$, when equality (3.5) is to be used instead of it.

Let us show that investigation of the structure of the velocity field downstream of the flow does not result in any supplementary boundary conditions which it would be necessary to satisfy for integrating Eq. (2.3). For $\xi \to +\infty$ the solution, which defines a chemically active mixture, must tend to the solution for an inviscid inert gas. However, all off-design modes of nozzle operation with an inert gas are specified by expansions which for $\xi \to +\infty$ asymptotically approach integral (2.9) with $a = a_2$ [6]. This implies that for considerable positive ξ the solution of Eq. (2.3) must pass in the neighborhood of the straight line $f = a_2\xi$. The last requirement is satisfied only when the asymptotic representation of function f contains two arbitrary constants.

The asymptotic properties of integral curves of Eq. (2.3) can be readily established by expressing function f in the form (2.10), where, clearly, $a = a_2$. The correction χ is specified by formula (2.11). From this for $\xi \to +\infty$ and $\eta \to -\infty$ we obtain

$$\chi = (-\eta)^{-\frac{a-1}{a}} G\left(\frac{a-1}{a}, \frac{\beta}{\alpha a}; -\eta\right)\left[C_1 \frac{\Gamma\left(\frac{2a-1}{a} - \frac{\beta}{\alpha a}\right)}{\Gamma\left(1 - \frac{\beta}{\alpha a}\right)} + C_2 \frac{\Gamma\left(\frac{1}{a} + \frac{\beta}{\alpha a}\right)}{\Gamma\left(\frac{1}{a}\right)}\right] + \ldots$$

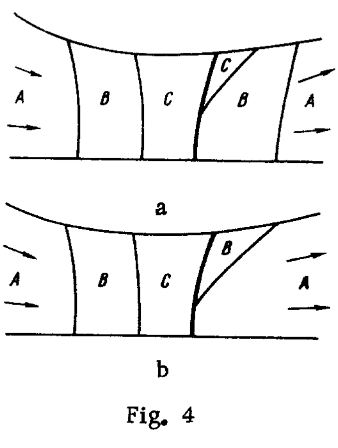

Fig. 4

where the dots denote exponentially small terms. Since for $a = a_2$ and any values of constants C_1 and C_2 the exponent

$$-\frac{a-1}{a} = \frac{1 - \sqrt{1 + 8/\nu}}{1 + \sqrt{1 + 8/\nu}} < 0$$

the second term in the right-hand part of formula (2.10) is considerably smaller than the first term. Hence both C_1 and C_2 are arbitrary. For the considered here solutions these constants are determined by numerical computation associated with the continuation of integral curves of Eq. (2.3), which are uniquely defined by the Cauchy formulas (4.5) and (4.6), into the region of considerable ξ. Curves presented in Fig. 2 relate to $\xi_s = 0.3,\ 0.5$ and 0.7.

For $f_1 = a_1\xi_s$ the derivative $df_2^{\cdot} / d\xi = 1 + a_2 + 2\beta / \alpha$ is independent of the position of the compression shock. Behind its front the gradients of the relaxing mixture parameters are comparatively great, hence the compression shock with the adjacent downstream region can be treated as a shock wave with incomplete dispersion. Incompletely dispersed shock waves in one-dimensional flows have a qualitatively very similar structure [3, 12, 13].

Velocity fields in flows with discontinuities are shown in Fig. 4, a and b, where the compression shock is shown by heavy line. The first of these modes is characterized by a fairly weak shock wave behind whose front the perturbed velocity is $-\sigma < v_{x2}(x_s, 0) < {}^1/_2 - \sigma$. In the second mode the compression shock intensity is considerably greater so that $v_{x2}(x_s, 0) < -\sigma$. Both these modes relate to inert gas flows in which the supersonic region occupies the whole of the Laval nozzle throat. Since for an inert gas $df_2 / d\xi = 1 + a_2$, hence in the flow region contiguous from behind to the discontinuity line gradients of the relaxing mixture parameters are greater than in the case of an inert gas. The difference is explained by the chemical transmutation of mixture elements in the incompletely dispersed shock wave.

We note in conclusion that for integrals $f = a\xi$ and $a = a_{1,2}$ function $h = a(\xi - \alpha)$. In conformity with the definition (2.8) of chemical affinity, we obtain from this

$$Q = -2ve_{12\infty}\alpha a \frac{d}{c^2} \neq 0$$

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NEW ANALYTIC SOLUTIONS OF THE WAVE EQUATION AND THE PROBLEM OF DIFFRACTION

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An earlier paper [1] gave the exact solutions for cylindrical and spherical waves, which made possible the solution of the problem of diffraction of waves due to a three-dimensional and a plane source. In the present paper the class of exact solutions is expanded significantly. The problem of diffraction of a wave due to a plane source by a semi-infinite plate is solved in a finite form.

1. We know [1] that if a solution of the wave equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}-\frac{\partial^2\Phi}{\partial t^2}=0 \tag{1.1}$$

is homogeneous in t and $r=\sqrt{x^2+y^2}$ of degree $-{}^1/_2$ and has the form $\Phi_{-1/2}(t, r, \theta)$, then $\Phi_{-1/2}(t+\alpha(t^2-r^2), r, \theta)$, where $\alpha=\mathrm{const}$ and $\theta=\operatorname{arctg}(y/x)$, also satisfies (1.1). On the other hand, the relation connecting the homogeneous solutions of the wave equation which have different degrees, is well known. In particular, if Φ_0 and Φ_n are solutions of (1.1) homogeneous in t and r of degrees 0 and n, respectively, and such that $(\Phi_n/t^n)|_{t=r}=\Phi_0|_{t=r}$, then they are connected by the following relation [2, 3]:

$$\Phi_n=\frac{(-1)^n 2^n n!}{(2n)!}(t^2-r^2)^{n+1/2}\frac{\partial^n}{\partial t^n}\frac{\Phi_0(r,t,\theta)}{\sqrt{t^2-r^2}} \tag{1.2}$$

Let us now set, in a purely formal way, the sum of the solutions of the wave equation, using the relation (1.2)